Supplementary appendices

B. Refinements of sequential equilibrium

In this Appendix, I describe two refinements of sequential equilibrium in the context of the bargaining game studied in this paper: the Intuitive Criterion and the undefeated equilibrium. I focus on the model of Section 5 where the real asset is only traded in the PM.

A typical signaling game has the following structure. There are two players: a sender of information and a receiver of information. In the context of the bargaining game in this paper, the sender is the buyer who makes an offer and the receiver is the seller who accepts or rejects the offer. The timing of the game is:

1. Nature draws a type t for the sender according to some (commonly known) probability distribution $\pi(t)$. Here, the set of types is $T = \{\ell, h\}$ and $\pi(\ell) = \pi_\ell$ and $\pi(h) = \pi_h$.

2. The sender (buyer) privately observes the type t, and he sends an offer o to the receiver (seller). Here, an offer is a triple $(q, d, \omega) \in \mathbb{R}^3_+$ where q is the output, d is the transfer of the real asset, and ω is the amount of real balances.

3. The receiver observes the offer o and takes an action r. Here, the set of actions is $\{Y, N\}$. If $r = Y$ then the offer is accepted; If $r = N$ then the offer is rejected.

The payoff of the buyer is $U^b(t, o, r) = -i\omega + [u(q) - \beta\kappa_t d - \omega] \mathbb{I}_{\{r=Y\}}$. The payoff of the seller is $U^s(t, o, r) = [-c(q) + \beta\kappa_t d + \omega] \mathbb{I}_{\{r=Y\}}$. After receiving the offer o, the seller forms a posterior probability assessment over the set of types of the buyer, $\lambda(t\,|o)$. The best response of the seller is

$$\text{BR}(\lambda, o) = \arg\max_{r\in\{Y,N\}} \sum_{t\in\{\ell,h\}} \lambda(t\,|o)\, U^s(t, o, r).$$

In the context of the bargaining game, the best response of the seller can be reexpressed as

$$\text{BR}(\lambda, o) = \arg\max_{r\in\{Y,N\}} \left\{[-c(q) + \beta\left[\lambda(h\,|o)\,\kappa_h + \lambda(\ell\,|o)\,\kappa_\ell\right] d + \omega]\, \mathbb{I}_{\{r=Y\}}\right\}.$$

I adopt the tie-breaking rule according to which $r = Y$ whenever $\text{BR}(\lambda, o) = \{Y, N\}$.

The equilibrium concept for the extensive game with imperfect information is that of sequential equilibrium. It is a pair of strategies and a belief system with the following properties. Strategies are sequentially rational: for each information set of each player i, the strategy of player i is a best response to the other player's strategy, given i's belief at that information set.

Let O denote a strategy for a buyer. It is a mapping from the set of types to the set of feasible offers. Let R denotes a strategy for a seller. It is a mapping from the set of feasible offers to the set $\{Y, N\}$. A (pure strategy) sequential equilibrium is a profile of strategies (O^*,R^*) and a seller's belief system, λ^*, such that the following is true.

1. For all $t \in T$, $\text{O}^*(t) \in \arg\max_{o'} U^b(t, o', \text{R}^*(o'))$

2. For all o, $\text{R}^*(o) \in \text{BR}(\lambda^*(o), o)$

3. λ^* satisfies Bayes' rule whenever possible (and is unconstrained for out-of-equilibrium offers)

In the context of the bargaining game studied in this paper, the buyer's strategy can be simplified by noticing the following. First, $U^b(t, o', \text{R}^*(o')) \leq 0$ for all o' such that $\text{R}^*(o') = \{N\}$. Second, from the tie-breaking rule, $U^b(t, o', \text{R}^*(o')) = 0$ and $\text{R}^*(o') = \{Y\}$ for $o' = (0, 0, 0)$. Hence, with no loss, the buyer can choose an offer among those that are accepted by sellers, i.e.,

$$\text{O}^*(t) \in \arg\max_{o'} U^b(t, o', Y) \quad \text{s.t.} \quad \{Y\} \in BR(\lambda^*(o'), o').$$

The buyer's problem becomes then (19)-(20).

The Intuitive Criterion

The Cho-Kreps (1987) refinement is based on the (intuitive) idea that out-of-equilibrium actions should never be attributed to a type who would not benefit from it under any circumstances. For a subset $K \subseteq T$, let $\text{BR}(K, o)$ denote the set of best responses for the seller to beliefs concentrated on K, i.e.,

$$\text{BR}(K, o) = \bigcup_{\{\lambda : \lambda(K) = 1\}} \text{BR}(\lambda, o).$$

Suppose $K = T = \{\ell, h\}$. Then,

$$\begin{aligned} BR(T, o) &= \{Y\} \quad \text{if} \quad -c(q) + \beta\kappa_\ell d + \omega > 0 \\ &= \{N\} \quad \text{if} \quad -c(q) + \beta d\kappa_h + \omega < 0 \\ &= \{Y, N\} \quad \text{otherwise.} \end{aligned}$$

Consider a proposed equilibrium where the payoff of a buyer of type t is denoted U_t^*. According to Cho and Kreps (1987, p.202), this proposed equilibrium fails the Intuitive Criterion if there exists an unsent offer o' such that:

1. $U_\ell^* > \max_{r \in BR(\{\ell,h\},o')} U^b(\ell, o', r)$
2. $U_h^* < \min_{r \in BR(\{h\},o')} U^b(h, o', r)$

According to the first requirement, the unsent offer o' would reduce the payoff of the $\ell-$type buyer compared to his equilibrium payoff irrespective of the inference the seller draws from o'. Consequently, the seller should attribute the offer o' to an $h-$type buyer. If he does so, the second requirement specifies that the $h-$type buyer should obtain a higher utility with o' compare to his equilibrium payoff.

In the bargaining game, the buyer's equilibrium payoff is bounded below by 0. Hence, the second requirement implies $\min_{r \in BR(\{h\},o')} U^b(h, o', r) > 0$, which requires $\{Y\} \in BR(\{h\}, o')$ (where I am making use of the tie-breaking rule). Since $\{Y\} \in BR(\{\ell, h\}, o')$, the first requirement becomes $U_\ell^* > U^b(\ell, o', Y)$. To summarize, a proposed equilibrium fails the Intuitive Criterion if there is an out-of-equilibrium offer that satisfies (61)-(63).

Undefeated sequential equilibria

Mailath, Okuno-Fujiwara and Postlewaite (1993) criticized the logical foundations of refinements based on forward induction (such as the Intuitive Criterion). They argue that it is difficult to interpret out-of-equilibrium messages as signals. For instance, consider a sequential equilibrium of the bargaining game that is pooling. It has been shown that an $h-$type buyer could make an out-of-equilibrium offer that would make him better-off and that would hurt an $\ell-$type buyer. (See Lemma 1.) By making such an offer the $h-$type buyer hopes to convince the seller that he is an $h-$type. But if the seller finds the Intuitive Criterion appealing he knows that the $h-$type buyer will alter his offer, and hence he should update his belief about the buyer's type if he does send the equilibrium message. "But if the (seller) does this, in the determination of whether a particular type might benefit from sending some disequilibrium message, the relevant comparison is not with the utility that he would receive in the proposed equilibrium but rather the utility that he would receive given that (the seller) is thinking in this way". (Mailath, Okuno-Fujiwara and Postlewaite, 1993, p.250.) They introduce a new refinement, based on the notion of *undefeated equilibrium*, that takes care of some of these concerns.

An equilibrium is composed of a strategy for buyers, $\textsc{o}$, that specifies an offer for each type, an acceptance rule for sellers, $\textsc{r}$, and a belief system for sellers, λ. According to Mailath, Okuno-Fujiwara and Postlewaite (1993, p.254, Definition 2) an equilibrium ($\textsc{o}$', $\textsc{r}$', λ') defeats ($\textsc{o}$,$\textsc{r}$, λ) if there exists an offer o' such that:

1. For all t, $\textsc{o}(t) \neq o'$ and $K \equiv \{t \in T \,|\textsc{o}'(t) = o'\} \neq \varnothing$
2. For all $t \in K$, $U^b\,[t, o', \textsc{r}'(o')] \geq U^b\,[t, \textsc{o}(t), \textsc{r}(\textsc{o}(t))]$ with a strict inequality for one t in K
3. $\lambda(t\,|o'\,) \neq p(t)\pi(t)/\sum_{t'} p(t')\pi(t')$ for at least one t in K where $p(t) = 1$ if $t \in K$ and $U^b\,[t, o', \textsc{r}'(o')] > U^b\,[t, \textsc{o}(t), \textsc{r}(\textsc{o}(t))]$ and $p(t) = 0$ if $t \notin K$.

So, for a sequential equilibrium to be defeated there must exist an out-of-equilibrium offer that is used in an alternative sequential equilibrium by a subset K of buyers' types (requirement 1). For all buyers with types in K, their payoff at the alternative equilibrium must be greater than the one at the proposed equilibrium with a strict inequality for at least one type (requirement 2). Finally, the belief system in the proposed equilibrium does not update sellers' prior belief conditional on the buyer's type being in K (requirement 3).

C. Undefeated monetary equilibria

The Intuitive Criterion is based on the simple idea that one should not attribute an out-of-equilibrium action to a type that would not benefit from it under any circumstances. While it is an intuitive refinement, its logic as well as some of the properties of the equilibria it selects have not stayed unchallenged.[41] Mailath, Okuno-Fujiwara and Postlewaite (1993) introduced an alternative refinement, *undefeated equilibria*, which addresses some of the shortcomings of refinements based on forward induction. In this section, I check the robustness of the results by investigating the conditions under which the equilibrium selected by the Intuitive Criterion is undefeated.[42] I focus on the model in Section 5 where the real asset is only traded in the PM.

The idea of an undefeated equilibrium is as follows. Consider a proposed sequential equilibrium and an out-of-equilibrium offer $\tilde{o}$. Suppose there is an alternative sequential equilibrium in which a subset of buyers' types choose $\tilde{o}$. Moreover, those buyers prefer the alternative equilibrium to the proposed equilibrium. The test requires seller's beliefs at that action in the original equilibrium to be consistent with the set of buyers who would benefit from the out-of-equilibrium offer $\tilde{o}$ (see the Appendix B for a formal definition). If the beliefs are not consistent, the second equilibrium *defeats* the proposed *equilibrium.*

In the following the attention will be restricted to symmetric equilibria in pure strategies.

Lemma 8 *The separating equilibrium that satisfies the Intuitive Criterion is the only undefeated equilibrium if*

$$\bar{U}_h^b \equiv -(1+i)\omega^p + u(q^p) - \beta\kappa_h d^p < U_h^b \equiv -(1+i)\omega_h + u(q_h) - \beta\kappa_h d_h, \tag{75}$$

where (q_h, d_h, ω_h) *is the solution to (24)-(26) and*

$$(q^p, d^p, \omega^p) = \arg\max_{\omega,q,d\leq A} \{-(1+i)\omega + u(q) - \beta\kappa_h d\} \tag{76}$$

$$s.t. \quad -c(q) + \beta\left(\pi_h\kappa_h + \pi_\ell\kappa_\ell\right)d + \omega = 0, \tag{77}$$

Conversely, if $\bar{U}_h^b > U_h^b$ *then there is an undefeated equilibrium and it is pooling.*

[41] One of the most problematic aspect of the equilibrium selected by the Intuitive Criterion might be its lack of continuity with respect to perturbations of the prior beliefs. In particular, the equilibrium obtained by taking the limit as π_ℓ goes to 0 does not approach the complete information equilibrium.

[42] As Mailath, Okuno-Fujiwara and Postlewaite (1993, p.265) put it,

> "[T]here is no reason that different refinements shouldn't be employed in the analysis of a single game. Various implausibilities may be exhibited in different equilibria of a game, and hence, considering different refinements of the equilibrium set for a single game is like looking at the game from different vantage points".

Proof. First, I establish that among the separating sequential equilibria, the only one that can be undefeated is the one that satisfies the Intuitive Criterion. Consider a sequential equilibrium that is separating and denote $\hat{U}_h^b$ the payoff of the $h-$type buyer at this equilibrium. The offer of the $\ell-$type buyer is his complete information offer, $(q_\ell, d_\ell, \omega_\ell)$. Suppose that the offer $(\hat{q}_h, \hat{d}_h, \hat{\omega}_h)$ of the $h-$type at the proposed equilibrium is different from (q_h, d_h, ω_h) that solves (24)-(26). Since (q_h, d_h, ω_h) is an offer made by the $h-$type only, and $\hat{U}_h^b < U_h^b$, then the proposed equilibrium is defeated by the unique separating equilibrium that satisfies the Intuitive Criterion.

Second, suppose (75) holds. Since, from (76)-(77), $\bar{U}_h^b$ is the highest payoff an $h-$type buyer can reach at a pooling equilibrium, any pooling equilibrium is defeated by the Pareto-efficient separating equilibrium.

Third, suppose that $\bar{U}_h^b > U_h^b$. The pooling equilibrium (q^p, d^p, ω^p) defeats the Pareto-efficient separating equilibrium. To see this, notice that the $h-$type buyer strictly prefers the pooling equilibrium. Moreover, the $\ell-$type buyer also prefers the pooling equilibrium, i.e.,

$$-(1+i)\omega^p + u(q^p) - \beta\kappa_\ell d^p > U(\kappa_\ell). \tag{78}$$

Indeed, from (77) $-c(q^p) + \beta\kappa_h d^p + \omega^p > 0$. Since the solution to (24)-(26) is such the seller's participation constraint and the $\ell-$type buyer incentive-compatibility constraint hold, $\bar{U}_h^b > U_h^b$ implies that the incentive-compatibility constraint is violated, and hence (78) holds. Finally, the pooling equilibrium (q^p, d^p, ω^p) is undefeated among pooling equilibria because the payoff of the $h-$type is maximized. Hence, any out-of-equilibrium offer that would correspond to a different pooling equilibrium can be attributed to an $\ell-$type buyer (so that the payoff of the $\ell-$type buyer associated with this out-of-equilibrium offer would be no greater than $U(\kappa_\ell)$). ■

Define the *lexicographically maximum* (Lex Max) pooling allocation as the solution to (76)-(77).[43] (Notice that the Lex Max pooling allocation corresponds to a sequential equilibrium if the payoff of the $\ell-$type buyer is at least equal to his complete information payoff.) The *lexicographically maximum sequential equilibrium* (LMSE) corresponds to the Lex Max pooling allocation if $\bar{U}_h^b \geq U_h^b$, and to the separating equilibrium given by the Intuitive Criterion otherwise.[44] The LMSE is intuitively appealing because it corresponds to the preferred sequential equilibrium of an $h-$type buyer. Lemma 8 shows that the LMSE is undefeated (Mailath

[43] Consider two sequential equilibria with the associated profile of payoffs for the buyers (u_h, u_ℓ) and (u'_h, u'_ℓ). The first equilibrium lexicographically dominates the second one if $u_h > u'_h$ or $u_h = u'_h$ and $u_\ell > u'_\ell$.

[44] A belief system consistent with the pooling outcome is as follows. For the equilibrium offer, the Bayes rule implies $\lambda(q^p, d^p, \omega^p) = \pi_h$. For all out-of-equilibrium offers that generate a payoff to $\ell-$type buyers greater than $U(\kappa_\ell)$ then $\lambda = 0$. For other out-of-equilibrium offers, $\lambda = \pi_h$.

et al., 1993, Theorem 1) and if it is completely separating, it is the only undefeated (pure strategy) sequential equilibrium (Mailath *et al.*, 1993, Theorem 2).

The determination of undefeated equilibria is illustrated in Figure 9 in the case without fiat money. The separating equilibrium is such that the $\ell-$type buyer gets his complete information payoff (the indifference curves U_ℓ^s and U_ℓ^b are tangent) and the trade in a match with an $h-$type buyers is such that both the participation constraint of the seller and the incentive-compatibility condition for the $\ell-$type bind (U_h^s and U_ℓ^b intersect). The Lex Max pooling allocation is at the tangency point between the indifference curve of the seller given his prior belief, $\bar{U}^s$, and the indifference curve of an $h-$type buyer. If the utility of the $h-$type buyer at separating equilibrium is greater than his utility at the pooling allocation, as in Figure 9, then the separating equilibrium is undefeated.

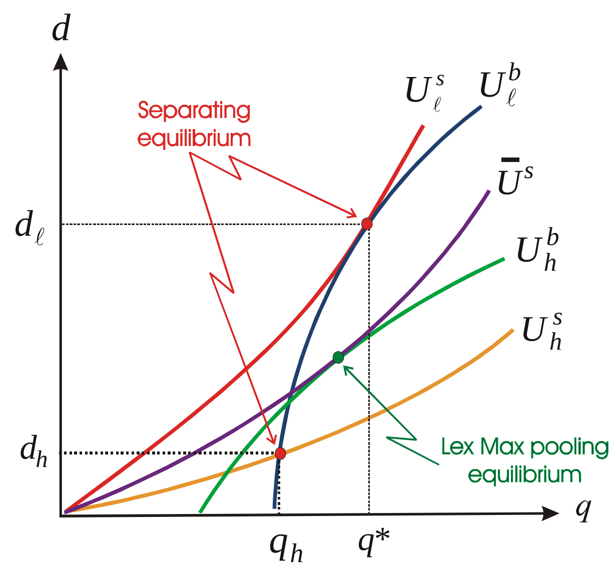

Figure 9: Undefeated equilibria

The next Lemma characterizes the Lex Max pooling allocation. Let $q_m(i)$ denote the value of q that solves $u'(q)/c'(q) = 1 + i$.

Lemma 9 *The Lex Max pooling allocation (q, d, ω) that solves (76)-(77) is such that:*

1. *If $i < \kappa_h/\bar{\kappa} - 1$ then $d = 0$, $q = q_m(i)$ and $\omega = c(q)$.*

2. *If $i = \kappa_h/\bar{\kappa} - 1$ then $q_m(i)$ and $\omega + \beta\bar{\kappa}d = c(q)$.*

3. *If* $i > \kappa_h/\bar{\kappa} - 1$ *then:*

   (a) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \geq 1+i$ *then* $d = A$, $q = q_m(i)$ *and* $\omega = c(q) - \beta\bar{\kappa}A$.

   (b) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \leq \kappa_h/\bar{\kappa}$ *then* $u'(q) = \frac{\kappa_h}{\bar{\kappa}}c'(q)$, $\omega = 0$ *and* $d = c(q)/\beta\bar{\kappa}$.

   (c) *If* $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \in (\kappa_h/\bar{\kappa}, 1+i)$ *then* $q = c^{-1}(\beta\bar{\kappa}A)$, $\omega = 0$ *and* $d = A$.

Proof. Assume the constraint $\omega \geq 0$ is not binding and substitute ω given by (77) into (76) to get

$$\bar{U}_h^b = \max_{q,0\leq d\leq A}\left\{u(q) - (1+i)c(q) + \beta d\left[(1+i)\bar{\kappa} - \kappa_h\right]\right\}$$

where $\bar{\kappa} = \pi_h\kappa_h + \pi_\ell\kappa_\ell$. If $i < \kappa_h/\bar{\kappa} - 1$ then $d = 0$, $u'(q)/c'(q) = 1+i$ and $\omega = c(q)$. This gives case 1 in the Lemma. If $i = \kappa_h/\bar{\kappa} - 1$ then $u'(q)/c'(q) = 1+i$ and $\omega + \beta\bar{\kappa}d = c(q)$ (the exact combination of ω and d is indeterminate.) This gives case 2 in the Lemma. If $i > \kappa_h/\bar{\kappa} - 1$ then $d = A$, $u'(q)/c'(q) = 1+i$ and $\omega = c(q) - \beta\bar{\kappa}A$ provided that $u'(q)/c'(q) \geq 1+i$ at $q = c^{-1}(\beta\bar{\kappa}A)$, which guarantees that $\omega \geq 0$. This gives case 3(a) in the Lemma. Assume next that $\omega \geq 0$ is binding. Then, from (76)-(77),

$$\bar{U}_h^b = \max_{0\leq d\leq A}\left[u\left[c^{-1}(\beta\bar{\kappa}d)\right] - \beta\kappa_h d\right].$$

If $d \leq A$ does not bind then $u'(q) = \frac{\kappa_h}{\bar{\kappa}}c'(q)$ and $d = c(q)/\beta\bar{\kappa}$. The constraint $d \leq A$ can then be reexpressed as $c(q) \leq \beta\bar{\kappa}A$ or $u'\left[c^{-1}(\beta\bar{\kappa}A)\right]/c'\left[c^{-1}(\beta\bar{\kappa}A)\right] \leq \kappa_h/\bar{\kappa}$. If $d \leq A$ binds then $q = c^{-1}(\beta\bar{\kappa}A)$. The constraint $\omega = 0$ requires $u'(q)/c'(q) \leq 1+i$. If $c(q) \leq \beta\bar{\kappa}A$ then the condition holds if $i \geq \kappa_h/\bar{\kappa} - 1$. If $d \leq A$ binds then $u'(q)/c'(q) < 1+i$ must be evaluated at $q = c^{-1}(\beta\bar{\kappa}A)$. This gives cases 3(b) and 3(c) in the Lemma.
■

The pooling allocation is such that fiat money is the only means of payment provided that the cost of real balances is sufficiently low, $i < \kappa_h/\bar{\kappa} - 1$. Intuitively, if the two types of buyers are pooled in equilibrium then the $h-$type buyer incurs a cost equal to $(\kappa_h - \bar{\kappa})/\bar{\kappa}$ per unit of the real asset he sells while the expected cost of holding real balances is i. In contrast, in the separating equilibrium both types of buyers use the real asset as means of payment. The quantities traded in the pooling monetary equilibrium are the same as the ones that buyers would trade in a separating monetary equilibrium provided that both buyers hold money.

Proposition 8 *There exists* $\bar{\pi}_h \in (0,1)$ *such that for all* $\pi_h < \bar{\pi}_h$ *the only undefeated equilibrium is the (separating) equilibrium that satisfies the Intuitive Criterion whereas for all* $\pi_h > \bar{\pi}_h$ *any undefeated equilibrium is pooling.*

Proof. From (24)-(26) U_h^b, the utility of an $h-$type buyer at the separating equilibrium, is independent of π_h. From Lemma 9, $\bar{U}_h^b$, the utility of an $h-$type buyer at the Lex Max pooling allocation, is a continuous and (weakly) increasing function of π_h. It is strictly increasing provided that $i > \kappa_h/\bar{\kappa} - 1$, or equivalently, $\pi_h > \frac{\kappa_h-(1+i)\kappa_\ell}{(1+i)(\kappa_h-\kappa_\ell)}$, and it is constant otherwise.

If $\pi_h = 1$ then (q^p, d^p, ω^p) solves (24) subject to (25) without (26) being imposed. From Lemma 4, (26) is binding at the separating equilibrium. Hence, $\bar{U}_h^b > U_h^b$.

If $\pi_h = 0$ then the problem (76)-(77) is analogous to (22) except from the fact that the objective of the $\ell-$type buyer is replaced by the objective of the $h-$type buyer. Hence,

$$-(1+i)\omega^p + u(q^p) - \beta\kappa_\ell d^p \leq U(\kappa_\ell)$$

and (26) holds at (q^p, d^p, ω^p), with a strict inequality if $d^p = 0$ since $d_\ell > 0$. Moreover,

$$0 = -c(q^p) + \beta\kappa_\ell d^p + \omega^p \leq -c(q^p) + \beta\kappa_h d^p + \omega^p,$$

with a strict inequality if $d^p > 0$. From Lemma 4, both (25) and (26) are binding at the optimum so that $\bar{U}_h^b < U_h^b$. Consequently, there is a unique $\bar{\pi}_h \in (0,1)$ such that $\bar{U}_h^b = U_h^b$; for all $\pi_h > \bar{\pi}_h$, $\bar{U}_h^b > U_h^b$; for all $\pi_h < \bar{\pi}_h$, $\bar{U}_h^b < U_h^b$. ■

The equilibrium that satisfies the Intuitive Criterion is the unique undefeated equilibrium provided that the fraction of $h-$type buyers is not too large. Intuitively, $h-$type buyers face a trade-off between the tightness of the seller's participation constraint and the tightness of the $\ell-$type buyer's incentive-compatibility constraint. If the fraction of $\ell-$type buyers is sufficiently small then $h-$type buyers cannot loosen much the seller's participation constraint by separating themselves from the $\ell-$type buyers, i.e., $\pi_\ell\kappa_\ell + \pi_h\kappa_h$ is close to κ_h. However, they would have to tighten significantly the incentive-compatibility condition to separate themselves from the $\ell-$type buyers. Hence, $h-$type buyers are happily pooled with $\ell-$type ones.

Next, I investigate how monetary policy affects the set of undefeated equilibria.

Proposition 9 *For all $i < \kappa_h/\bar{\kappa} - 1$, the unique undefeated equilibrium is the unique (separating) equilibrium that satisfies the Intuitive Criterion.*

Proof. Consider the pooling allocation in (76)-(77). From Lemma 9, for all $i < \kappa_h/\bar{\kappa} - 1$ then $d = 0$ and

$$\bar{U}_h^b(i) = -(1+i)c[q_m(i)] + u[q_m(i)].$$

Moreover, $d\bar{U}_h^b/di = -c\,[q_m(i)] < 0$. Consider next the Pareto-efficient separating equilibrium. For all $i < i_2(A)$, defined in the proof of Proposition 2,

$$U_h^b(i) = -(1+i)c\,[q_m(i)] + u\,[q_m(i)] + i\beta\kappa_h d_h(i),$$

and $U_h^b(i) = U_h^b[i_2(A)]$ for all $i > i_2(A)$. Since $d_h(i) > 0$ for all $i > 0$ then $U_h^b(i) > \bar{U}_h^b(i)$ for all $i < \kappa_h/\bar{\kappa} - 1$ and the unique undefeated equilibrium is the separating equilibrium. ■

Provided that the cost of holding real balances is not too high, the unique undefeated equilibrium is separating. Hence, the results in Section 5 are robust to the adoption of the alternative refinement from Mailath, Okuno-Fujiwara and Postlewaite (1993).

A second insights from Proposition 9 is that monetary policy can affect the nature of the (undefeated) equilibrium. Suppose that i is sufficiently large so that no monetary equilibrium exists. From Proposition 8, provided that π_h is sufficiently large, any undefeated equilibrium is pooling. Consider a reduction of the cost of holding real balances. From Proposition 9, provided that i is sufficiently low, the unique undefeated equilibrium is separating. Hence, the reduction of the money growth rate changes the nature of the equilibrium from pooling to separating.[45]

[45] A similar result is found in Jafarey and Rupert (2001) where the nonmonetary equilibrium is always pooling while the monetary equilibrium can be separating.

D. Signaling with more than two types

This appendix shows the robustness of the results if one allows for more than two states for the dividend of the real asset. In particular, there is a unique equilibrium of the bargaining game and it is separating. I consider the version of the model in Section 5 where the real asset is only traded in the PM. I describe the case where the set of buyers' is $T = \{1, ..., K\}$ with $K \geq 3$ and K is finite. The dividend in the state k is κ_k with $0 < \kappa_1 < \kappa_2 < ... < \kappa_K$. The measure of buyers of type k is π_k.

In the PM the seller observes the offer (q, d, ω) made by the buyer, and he forms a belief about the buyer's type. Denote $\lambda(q, d, \omega)$ the probability measure of the buyer's type κ conditional on the offer (q, d, ω) being made. Formally, $\Pr[k \in S] = \int \mathbb{I}_S d\lambda(q, d, \omega)$ where $\mathbb{I}_S(k)$ is an indicator function that is equal to 1 if $k \in S$.

Consider a sequential equilibrium and let U_k^b denote the payoff of a buyer of type κ_k. The proposed equilibrium fails the Intuitive Criterion (Cho and Kreps, 1987, p.202) if there is an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\omega})$ and a set of types $S \subset T$ such that

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_k\tilde{d} \quad < \quad U_k^b \quad \forall k \in S \tag{79}$$

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_k\tilde{d} \quad > \quad U_k^b \quad \text{for some } k \in T \backslash S \tag{80}$$

$$-c(\tilde{q}) + \tilde{\omega} + \mathbb{E}_{\tilde{\lambda}}[\kappa_k]\beta\tilde{d} \quad \geq \quad 0 \quad \forall \tilde{\lambda} : \text{supp}(\tilde{\lambda}) \subseteq T \backslash S. \tag{81}$$

According to (79), the unsent offer makes buyers with types included in S strictly worse off compared to their equilibrium payoff. According to (80), it makes at least one buyer with type included in $T \backslash S$ strictly better off. According to (81) the offer is acceptable for any belief system that puts no weight on the types in S.

The next lemma shows that there is no offer involving a transfer of the real asset that is pooling.

Lemma 10 *In any equilibrium, there is no pooling offer such that* $d > 0$.

Proof. Suppose there is an equilibrium where a subset $\bar{T} \subseteq T$ of buyers' types (with at least two distinct elements) make the same offer $(\bar{q}, \bar{d}, \bar{\omega})$ with $\bar{d} > 0$. Hence, the equilibrium payoffs are

$$U_k^b \equiv -(1+i)\bar{\omega} + u(\bar{q}) - \beta\kappa_k\bar{d}, \quad \forall k \in \bar{T}.$$

This offer satisfies the seller's participation constraint. Hence,

$$-c(\bar{q}) + \mathbb{E}_{\lambda(\bar{q},\bar{d},\bar{\omega})}[\kappa]\beta\bar{d} + \bar{\omega} \geq 0, \tag{82}$$

where $\lambda(\bar{q}, \bar{d}, \bar{\omega})$ is determined by Bayes' rule.[46] Let $k = \max \bar{T}$ and $k' = \max \bar{T}\backslash\{k\}$. Suppose that a $k-$type buyer deviates and offers $(\tilde{q}, \tilde{d}, \tilde{\omega})$ such that $\tilde{\omega} = \bar{\omega}$, $\tilde{d} = \bar{d} - \varepsilon$ where $\varepsilon \in \left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k\right)$, and

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_{k'}\tilde{d} \quad < \quad U^b_{k'} \tag{83}$$

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_{k}\tilde{d} \quad > \quad U^b_{k}, \tag{84}$$

or, equivalently,

$$\kappa_{k'} < \frac{u(\bar{q}) - u(\tilde{q})}{\beta\varepsilon} < \kappa_k. \tag{85}$$

First, I establish that the set of offers $(\tilde{q}, \tilde{d}, \tilde{\omega})$ that satisfy the conditions above is not empty. Since, from Bayes' rule $\mathbb{E}_{\lambda(\bar{q},\bar{d},\bar{\omega})}[\kappa] < \kappa_k$, (82) implies $c(\bar{q}) < \beta\kappa_k\bar{d} + \bar{\omega}$ and $\left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k\right)$ is non-empty. The requirement $U^b_k \geq 0$ (in any equilibrium the buyers' payoffs are nonnegative since $(q, d, \omega) = (0, 0, 0)$ is always available) implies $u(\bar{q}) - \beta\kappa_k\bar{d} - \bar{\omega} \geq 0$ and hence

$$\frac{u(\bar{q})}{\beta\kappa_k} - \varepsilon \geq \bar{d} + \frac{\bar{\omega}}{\beta\kappa_k} - \varepsilon > 0$$

(since $\varepsilon < \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k$.) So for any $\varepsilon \in \left(0, \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k\right)$ there is a $\tilde{q} \geq 0$ satisfying

$$u(\bar{q}) - \beta\varepsilon\kappa_k < u(\tilde{q}) < u(\bar{q}) - \beta\varepsilon\kappa_{k'}$$

and hence (85).

Second, I show that

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_{k''}\tilde{d} < U^b_{k''} \quad \forall k'' < k'. \tag{86}$$

By incentive compatibility,

$$U^b_{k''} \geq -(1+i)\bar{\omega} + u(\bar{q}) - \beta\kappa_{k''}\bar{d}.$$

From (85), $u(\bar{q}) > u(\tilde{q}) + \beta\varepsilon\kappa_k > u(\tilde{q}) + \beta\varepsilon\kappa_{k''}$. Since $\tilde{\omega} = \bar{\omega}$ and $\tilde{d} = \bar{d} - \varepsilon$,

$$-(1+i)\tilde{\omega} + u(\tilde{q}) - \beta\kappa_{k''}\tilde{d} < -(1+i)\bar{\omega} + u(\bar{q}) - \beta\kappa_{k''}\bar{d} \leq U^b_{k''}.$$

This proves (86).

Finally, I show that any offer $(\tilde{q}, \tilde{d}, \tilde{\omega})$ disqualifies the proposed equilibrium according to the Intuitive Criterion. From (83) and (86), the set of types S such that (79) is true is $S \supseteq \{\kappa_{k''} : k'' \leq k'\}$. From

[46] In any equilibrium buyers make an acceptable offer. Indeed, their payoffs are bounded away from 0 since they can always achieve $\max_{q,\omega}[-(1+i)\omega + u(q)]$ s.t. $c(q) = \omega$.

(84), the condition (80) is satisfied for the buyer's type k. The condition $\varepsilon < \bar{d} + [\bar{\omega} - c(\bar{q})] / \beta\kappa_k$ implies $c(\bar{q}) < \beta\kappa_k \left(\bar{d} - \varepsilon\right) + \bar{\omega}$. From (85), $\bar{q} > \tilde{q}$. So,

$$c(\tilde{q}) < \beta\kappa_k\tilde{d} + \bar{\omega} \leq \beta\mathbb{E}_{\tilde{\lambda}(\tilde{q},\tilde{d},\tilde{\omega})}[\kappa]\,\tilde{d} + \bar{\omega}$$

for any $\tilde{\lambda}$ such that $\text{supp}\tilde{\lambda}(\tilde{q},\tilde{d},\tilde{\omega}) \subseteq T\backslash S$, since $T\backslash S \subseteq \{k, k+1, ..., K\}$. Consequently, (81) is also satisfied. ■

A buyer reveals his type through his equilibrium offer (unless $d = 0$ in which case the buyer's private information is irrelevant for the seller's payoff). Let (q_k, ω_k, d_k) indicate the offer made by a buyer of type k. The next Lemma shows that the quantity of the asset transferred to the seller is a non-increasing function of the buyer's type. The proof of this result is based on incentive compatibility.

Lemma 11 *If $k > k'$ then $d_k \leq d_{k'}$.*

Proof. Incentive-compatibility requires that for any two types k and k',

$$U_k^b = -(1+i)\omega_k + u(q_k) - \beta\kappa_k d_k \geq -(1+i)\omega_{k'} + u(q_{k'}) - \beta\kappa_k d_{k'}. \tag{87}$$

From (87), and interchanging k and k', one can show that

$$\kappa_{k'}(d_k - d_{k'}) \geq \kappa_k(d_k - d_{k'}).$$

Hence, if $k > k'$ then $d_k \leq d_{k'}$. ■

So, the result according to which buyers with a high type spend a smaller fraction of their real asset in the PM than buyers with a low type is robust across mechanisms. The next proposition characterizes the buyers' equilibrium offers and payoffs.

Proposition 10 *The equilibrium offer of a κ_1-buyer is the solution to*

$$U_1^b = \max_{(q,d,\omega)} \{-(1+i)\omega + u(q) - \beta\kappa_1 d\} \tag{88}$$

$$s.t. \quad -c(q) + \omega + \beta\kappa_1 d = 0. \tag{89}$$

The equilibrium offer of a κ_k-buyer, for $k \in \{2, ..., K\}$, solves

$$U_k^b = \max_{(q,d,\omega)} \{-(1+i)\omega + u(q) - \beta\kappa_k d\} \tag{90}$$

$$s.t. \quad -c(q) + \omega + \beta\kappa_k d \geq 0 \tag{91}$$

$$-(1+i)\omega + u(q) - \beta\kappa_{k-1} d \leq U_{k-1}^b. \tag{92}$$

Proof. The offers $\{(q_k, d_k, \omega_k)\}_{k=1}^{K}$ must be incentive-compatible, i.e.,

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k'}d_k \leq U_{k'}^b \tag{93}$$

for all $k, k' \in T$. First, I establish that (92) implies that the incentive-compatibility condition (93) holds for all $k' < k$ and $k \geq 2$. From (92),

$$U_k^b \equiv -(1+i)\omega_k + u(q_k) - \beta\kappa_k d_k \leq U_{k-1}^b - \beta d_k(\kappa_k - \kappa_{k-1}),$$

and, by successive iterations,

$$U_k^b \leq U_{k-n}^b - \sum_{t=k-n+1}^{k} \beta d_t(\kappa_t - \kappa_{t-1}), \qquad \forall n = 1, .., k-1.$$

By definition of U_k^b,

$$\begin{aligned} -(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k &= U_k^b - \beta\kappa_{k-n}d_k + \beta\kappa_k d_k \\ &\leq U_{k-n}^b - \sum_{t=k-n+1}^{k} \beta d_t(\kappa_t - \kappa_{t-1}) - \beta\kappa_{k-n}d_k + \beta\kappa_k d_k. \end{aligned}$$

Rearrange the sum on the right-hand side, and use an appropriate change of variable to obtain

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b + \sum_{t=k-n+1}^{k} \beta(d_t - d_{t-1})\kappa_{t-1} - \beta\kappa_{k-n}(d_k - d_{k-n}).$$

Using the fact that $d_k - d_{k-n} = \sum_{t=k-n+1}^{k}(d_t - d_{t-1})$, the inequality becomes

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b + \sum_{t=k-n+1}^{k} \beta(d_t - d_{t-1})(\kappa_{t-1} - \kappa_{k-n}).$$

From Lemma 11, $(d_t - d_{t-1})(\kappa_{t-1} - \kappa_{k-n}) \leq 0$ for all $t \in \{k-n+1, ..., k\}$. Hence, $-(1+i)\omega_k + u(q_k) - \beta\kappa_{k-n}d_k \leq U_{k-n}^b$ for all $n = 1, .., k-1$.

Next, I prove that the incentive-compatibility condition (93) holds for all $k' > k$. The proof is by induction. Suppose

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k+n}d_k \leq U_{k+n} \tag{94}$$

for $n > 0$. This inequality holds at $n = 0$. From (91),

$$-c(q_k) + \omega_k + \beta\kappa_{k+n+1}d_k > -c(q_k) + \omega_k + \beta\kappa_k d_k \geq 0. \tag{95}$$

From (94) and (95), (q_k, d_k, ω_k) satisfies (91)-(92) when k is replaced by $k+n+1$. Hence, one can deduce from (90) that

$$-(1+i)\omega_k + u(q_k) - \beta\kappa_{k+n+1}d_k \leq U^b_{k+n+1}.$$

So, I have checked that the incentive-compatibility conditions hold for all k, k'. To show that (q_1, ω_1, d_1) is the only possible offer for a κ_1-buyer notice from Lemma 10 that a buyer with the lowest type cannot do better than his complete information payoff and this payoff is always achievable for any belief system λ (since $\mathbb{E}_\lambda[\kappa] \geq \kappa_1$).

Next, I show that the solution (q_k, d_k, ω_k) to (90)-(92) for $k \geq 2$ is the only one that can satisfy the Intuitive Criterion. Suppose that the equilibrium payoff of a $k-$buyer is $\tilde{U}_k < U^b_k$. Replace U^b_{k-1} in (92) by $U^b_{k-1} - \varepsilon$ for some $\varepsilon > 0$ and denote

$$(q^\varepsilon_k, d^\varepsilon_k, \omega^\varepsilon_k) = \arg\max_{(q,d,\omega)} \{-(1+i)\omega + u(q) - \beta\kappa_k d\}$$

$$\text{s.t.} \quad -c(q) + \omega + \beta\kappa_k d \geq 0$$

$$-(1+i)\omega + u(q) - \beta\kappa_{k-1}d \leq U^b_{k-1} - \varepsilon.$$

Denote U^ε_k the value to this problem. Since U^ε_k is continuous in ε there exists $\varepsilon > 0$ such that $\tilde{U}_k < U^\varepsilon_k$. By an argument analogous to the one above,

$$-(1+i)\omega^\varepsilon_k + u(q^\varepsilon_k) - \beta\kappa_{k-n}d^\varepsilon_k < U^b_{k-n}, \quad \forall n = 1, .., k-1.$$

Moreover, if the support of the seller's belief is restricted to $\{k, .., K\}$ then $\mathbb{E}_\lambda[\kappa] \geq \kappa_k$ and, from (91), $(q^\varepsilon_k, d^\varepsilon_k, \omega^\varepsilon_k)$ is acceptable. So the proposed equilibrium violates the Intuitive Criterion.

Finally, I construct a belief system that generates $\{(q_k, d_k, \omega_k)\}_{k=1}^K$ as the solution to the buyers' problems. The belief system is such that $\int \mathbb{I}_{\{k\}} d\lambda(q_k, d_k, \omega_k) = 1$ from Bayes' rule. The beliefs for out-of-equilibrium offers are specified as follows. All out-of-equilibrium offers that generate a payoff strictly greater than U^b_1 to a type-1 buyer are attributed to a κ_1-buyer. By construction these offers are rejected by sellers. Among the remaining offers, all out-of-equilibrium offers that generate a payoff strictly greater than U^b_2 to a type-2 buyer are attributed to a type-2 buyer. These offers satisfy (92) for $k = 2$. Hence, they must violate (91) and, as a consequence, they are rejected by sellers. And so on. The out-of-equilibrium offers that make all players worse-off are attributed to the highest type. ■

The buyer with the lowest type makes his complete information offer. Buyers of type $k > 1$ maximize their expected utility subject to the participation constraint of sellers and the incentive-compatibility condition of buyers of type $k-1$.

The problem (90)-(92) is formally identical to (24)-(26). One can make use of Lemma 4 to provide the following characterization of the equilibrium offers.

Lemma 12 *For all $i > 0$ and all $k \in \{2, ..., K\}$, there is a unique solution (q_k, d_k, ω_k) to (90)-(92) and it solves:*

$$\omega_k = \frac{\kappa_k \left\{ \left[u\left(q_k\right) - \frac{\kappa_\ell}{\kappa_h} c(q_k) \right] - U^b_{k-1} \right\}}{(1+i)\kappa_k - \kappa_{k-1}} \tag{96}$$

$$d_k = \frac{U^b_{k-1} - \left[u\left(q_k\right) - (1+i)\, c(q_k) \right]}{\left[(1+i)\kappa_k - \kappa_{k-1}\right] \beta} \tag{97}$$

and

$$u'\left(q_k\right) - (1+i)\, c'(q_k) \leq 0 \quad \text{“} = \text{”} \quad \text{if } \omega_k > 0. \tag{98}$$

Both the seller's participation constraint (91) and the buyer's incentive-compatibility condition (92) are binding.

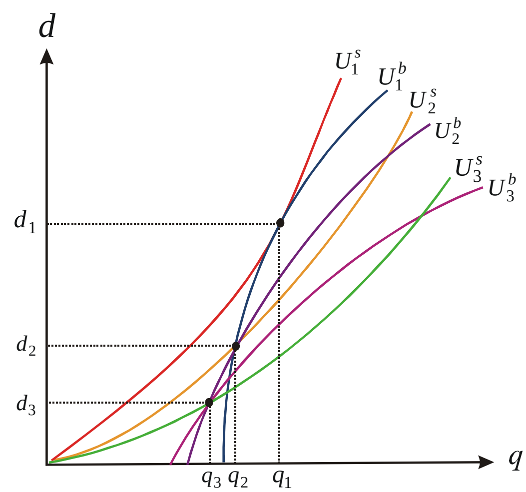

Figure 10: Equilibrium offers (economy without fiat money)

The determination of the equilibrium offers is illustrated in Figure 10 in the case without fiat money. The indifference curve of a seller matched with a buyer of type k is denoted U^s_k. The indifference curve of a buyer of type k is denoted U_k^b. The sellers' indifference curves go through the origin since their participation constraints are binding. Assuming the constraint $d \leq A$ is not binding, the terms of trade (q_1, d_1) are determined at the tangency of the seller's indifference curve and the buyer's indifference curve. In matches where the buyer's type is $k > 1$ then the terms of trade (q_k, d_k) are at the intersection of the seller's indifference curve, U_k^s, and the indifference curve of the buyer of type $k-1$, U_{k-1}^b.

E. Asset pricing under symmetric information

In order to isolate the role of private information, I analyze in this Appendix the economy with symmetric information: buyers and sellers in the PM have the same information about the future dividend of the asset.

Complete information

Consider a match in the PM between a buyer and a seller. The buyer holds ω real balances (expressed in terms of their discounted value in the next AM) and a units of capital. The future dividend of each unit of capital is κ and it is common knowledge in the match. The strategy of a buyer in the PM is a triple $[q(\omega,a,\kappa), d(\omega,a,\kappa), \tau(\omega,a,\kappa)]$ solution to

$$[q(\omega,a,\kappa), d(\omega,a,\kappa), \tau(\omega,a,\kappa)] = \arg\max_{q,\tau,d}\left[u(q) - \beta\kappa d - \frac{\beta}{\gamma}\tau\right] \tag{99}$$

$$\text{s.t.} \quad -c(q) + \beta\kappa d + \frac{\beta}{\gamma}\tau \geq 0, \tag{100}$$

$$\text{s.t.} \quad \frac{\beta}{\gamma}\tau \leq \omega, \quad d \leq a. \tag{101}$$

If $\beta\kappa a + \omega \geq c(q^*)$ then the solution to (99)-(101) is $q = q^*$ and $\beta\kappa d + \omega = c(q^*)$. Otherwise, $q = c^{-1}(\beta\kappa a + \omega)$, $d = \kappa a$ and $\beta\tau/\gamma = \omega$. Let denote $\hat{\mathcal{S}}(\beta\kappa a + \omega) = u(q) - c(q)$ the buyer's surplus where $q = \min\left[q^*, c^{-1}(\beta\kappa a + \omega)\right]$.

The buyer's portfolio decision in the AM is:

$$\max_{\omega\geq 0, a\geq 0}\left\{-i\omega - (\phi - \beta\bar{\kappa})\,a + \pi_h\hat{\mathcal{S}}(\omega + \beta\kappa_h a) + \pi_\ell\hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)\right\}, \tag{102}$$

where $\phi - \beta\bar{\kappa}$ in (102) represents the cost of investing in capital: it is equal to the price of capital in the AM minus the discounted expected dividend in the subsequent AM.

The clearing condition for the market for capital goods requires the fixed capital stock to be held by buyers,

$$\int_{j\in\mathcal{B}} a(j)dj = A. \tag{103}$$

(Recall that sellers cannot produce in the AM and hence cannot acquire capital.)

An equilibrium is a list of portfolios, a profile of buyers' strategies in the PM, and the price of capital that satisfy (99)-(103).

The following lemma characterizes the buyer's portfolio choice.

Lemma 13 *Assume $\phi > \beta\bar{\kappa}$. If $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell \neq i$ or $\pi_\ell \hat{\mathcal{S}}' [\kappa_\ell c(q^*)/\kappa_h] < i$ then the buyer's problem (102) admits a unique solution. It satisfies*

$$-i + \pi_h \hat{\mathcal{S}}'(\omega + \beta\kappa_h a) + \pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) \leq 0 \quad \text{“} = \text{”} \quad \text{if } \omega > 0 \tag{104}$$

$$-(\phi - \beta\bar{\kappa}) + \pi_h \beta\kappa_h \hat{\mathcal{S}}'(\omega + \beta\kappa_h a) + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) \leq 0 \quad \text{“} = \text{”} \quad \text{if } a > 0. \tag{105}$$

If $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell = i$ and $\pi_\ell \hat{\mathcal{S}}' [\kappa_\ell c(q^*)/\kappa_h] \geq i$ then any (ω, a) such that $\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = i$ and $\omega + \beta\kappa_h a \geq c(q^*)$ is solution to (102). If $\phi = \beta\bar{\kappa}$ then any $(\omega, a) \in \{0\} \times [c(q^*)/\beta\kappa_\ell, \infty)$ is solution to (102). Finally, if $\phi < \beta\bar{\kappa}$ then there is no solution to (102).

Proof. Since $\hat{\mathcal{S}}'(\omega + \beta\kappa a) = u'(q)/c'(q) - 1$, where $q = \min\left[q^*, c^{-1}(\omega + \beta\kappa a)\right]$, then $\hat{\mathcal{S}}$ is concave and the buyer's problem (102) is concave as well. The first-order conditions (104) and (105) are then necessary and sufficient. Three cases are distinguished.

(i) $\phi > \beta\bar{\kappa}$. The solution to (102) cannot be such that $\omega + \beta\kappa_\ell a > c(q^*)$. Indeed, if $\omega + \beta\kappa_\ell a > c(q^*)$ then $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = u'(q^*)/c'(q^*) - 1 = 0$. But then (104)-(105) imply $\omega = a = 0$. A contradiction. So, one can restrict (ω, a) to the compact set $\{(\omega, a) \in \mathbb{R}_{2+} : \omega + \beta\kappa_\ell a \leq c(q^*)\}$ and from the Theorem of the Maximum a solution to (102) exists. Next, I show that the problem (102) is strictly jointly concave for all (ω, a) such that $\omega + \beta\kappa_h a < c(q^*)$. The Hessian matrix associated with (102) is

$$\mathbb{H} = \begin{pmatrix} \pi_h \hat{\mathcal{S}}''_h + \pi_\ell \hat{\mathcal{S}}''_\ell & \pi_h \beta\kappa_h \hat{\mathcal{S}}''_h + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}''_\ell \\ \pi_h \beta\kappa_h \hat{\mathcal{S}}''_h + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}''_\ell & \pi_h (\beta\kappa_h)^2 \hat{\mathcal{S}}''_h + \pi_\ell (\beta\kappa_\ell)^2 \hat{\mathcal{S}}''_\ell \end{pmatrix}$$

where $\hat{\mathcal{S}}''_h \equiv \hat{\mathcal{S}}''(\omega + \beta\kappa_h a)$ and $\hat{\mathcal{S}}''_\ell \equiv \hat{\mathcal{S}}''(\omega + \beta\kappa_\ell a)$. For all (ω, a) such that $\beta\kappa_h a + \omega \leq c(q^*)$, $\hat{\mathcal{S}}''_\ell < 0$ and $\hat{\mathcal{S}}''_h < 0$ and

$$|\mathbb{H}| = (\kappa_h - \kappa_\ell)^2 \pi_h \pi_\ell \beta^2 \hat{\mathcal{S}}''_\ell \hat{\mathcal{S}}''_h < 0.$$

So, $\mathbb{H}$ is negative definite and any solution to (104)-(105) such that $\omega + \beta\kappa_h a < c(q^*)$ corresponds to a strict local maximum and hence, from the concavity of the objective, it corresponds to the global maximum.

Suppose next that the solution is such that $\omega + \beta\kappa_h a \geq c(q^*)$. From (104)-(105),

$$\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = \min\left[i, \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}\right],$$

with $\omega = 0$ if $i > \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$ and $a = 0$ if $i < \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$. So the solution is unique provided that $i \neq \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$. If $i = \frac{(\phi - \beta\bar{\kappa})}{\beta\kappa_\ell}$ then any pair (ω, a) such that $\omega + \beta\kappa_h a \geq c(q^*)$ and $\pi_\ell \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = i$ is solution to (102). For such pairs to exist, $\pi_\ell \hat{\mathcal{S}}'\left[\frac{\kappa_\ell}{\kappa_h} c(q^*)\right] \geq i$.

(ii) $\phi = \beta\bar{\kappa}$. The first-order condition for a requires $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = \hat{\mathcal{S}}'(\omega + \beta\kappa_\ell a) = 0$ and hence any $a \geq [c(q^*) - \omega] / \beta\kappa_\ell$ is part of a solution. But then (104) implies $\omega = 0$.

(iii) $\phi < \beta\bar{\kappa}$ then the first-order condition for a admits no solution. ■

The following proposition establishes the existence of an equilibrium and the uniqueness of the price of capital and the allocation of the PM output conditional on the realization of κ.[47] Denote q_ℓ and q_h the output levels in the PM when $\kappa = \kappa_\ell$ and $\kappa = \kappa_h$, respectively.

Proposition 11 *There exists an equilibrium and (ϕ, q_ℓ, q_h) is uniquely determined. If $A \geq c(q^*)/\beta\kappa_\ell$ then the equilibrium is nonmonetary and $\phi = \beta\bar{\kappa}$. If $A < c(q^*)/\beta\kappa_\ell$ then $\phi > \beta\bar{\kappa}$ and there is $i_0 > 0$ such that for all $i < i_0$ an equilibrium is monetary.*

Proof. The proof proceeds in four parts. It first characterizes the correspondence

$$A^d \equiv \left\{ \int_{j \in \mathcal{B}} a(j)dj : a(j) \text{ solution to (102)} \right\}.$$

Second, it shows that ϕ is uniquely determined. Third, it establishes that the PM output levels, q_ℓ and q_h, are unique. Finally, it determines the conditions fiat money to be valued.

(i) Existence. Consider first the case $\phi > \beta\bar{\kappa}$. As shown in the proof of Lemma 13 (Part (i)), any solution (ω, a) to (102) lies in the compact set $[0, c(q^*)] \times [0, c(q^*)/\beta\kappa_\ell]$. Since the objective in (102) is continuous, the Theorem of the Maximum guarantees that $A^d(\phi)$ is nonempty and upper-hemi continuous. Since the objective in (102) is concave, $A^d(\phi)$ is convex-valued. From Lemma 13, $A^d(\beta\bar{\kappa}) = [c(q^*)/\beta\kappa_\ell, \infty)$. From (104) and (105), it can be checked that $A^d(\phi) = \{0\}$ for all $\phi > \beta\bar{\kappa} + i\beta\kappa_h$ and $A_d(\phi) = \{a\}$ where a solves

$$\pi_h \beta\kappa_h \hat{\mathcal{S}}'(\beta\kappa_h a) + \pi_\ell \beta\kappa_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell a) = \phi - \beta\bar{\kappa}$$

for all $\phi < \beta\bar{\kappa} + i\beta\kappa_\ell$ (since $\omega = 0$). Moreover, $a \to c(q^*)/\beta\kappa_\ell$ as $\phi \to \beta\bar{\kappa}$. Hence, from this characterization of $A^d(\phi)$, there is a $\phi \in [\beta\bar{\kappa}, \beta\bar{\kappa} + i\beta\kappa_\ell]$ such that $A \in A^d(\phi)$.

(ii) Uniqueness. In order to prove that ϕ is uniquely determined, I show that any selection from A^d is decreasing in ϕ: if $a_1 \in A^d(\phi_1)$ and $a_2 \in A^d(\phi_2)$ for $\phi_2 > \phi_1$ then $a_2 < a_1$ unless $a_2 = a_1 = 0$. Consider

[47] Buyers' portfolios are not always uniquely determined. If $(\phi - \beta\bar{\kappa}) / \beta\kappa_\ell = i$, and provided that $\hat{\mathcal{S}}'(\omega + \beta\kappa_h a) = 0$, real balances and capital are perfect substitutes. If $\phi = \beta\bar{\kappa}$ then buyers hold any quantity of capital above the level that satiates their liquidity needs in the PM, $\hat{\mathcal{S}}' = 0$, and they hold no real balances.

$\phi_2 > \phi_1$, and the associated portfolio choices (ω_1, a_1) and (ω_2, a_2). By revealed preferences,

$$\begin{aligned} -\phi_1 a_1 + \Psi(\omega_1, a_1) &\geq -\phi_1 a_2 + \Psi(\omega_2, a_2) \\ -\phi_2 a_2 + \Psi(\omega_2, a_2) &\geq -\phi_2 a_1 + \Psi(\omega_1, a_1), \end{aligned}$$

where $\Psi(\omega, a) \equiv -i\omega + \beta\bar{\kappa}a + \pi_h \hat{\mathcal{S}}(\omega + \beta\kappa_h a) + \pi_\ell \hat{\mathcal{S}}(\omega + \beta\kappa_\ell a)$. These last two inequalities yield

$$\phi_1(a_1 - a_2) \leq \Psi(\omega_1, a_1) - \Psi(\omega_2, a_2) \leq \phi_2 (a_1 - a_2).$$

Since $\phi_2 > \phi_1$ then $a_1 \geq a_2$. Suppose $a_1 = a_2 > 0$. From (105), $\omega_2 < \omega_1$ (where I have used that $\hat{\mathcal{S}}' < 0$ if $\omega + \beta\kappa a < c(q^*)$). But then, from (104), $\omega_1 = 0$. A contradiction.

(iii) The allocation (q_ℓ, q_h). From (i) and (ii), there exists a unique $\phi \geq \beta\bar{\kappa}$ such that $A \in A^d(\phi)$. If $A \geq c(q^*)/\beta\kappa_\ell$ then $\phi = \beta\bar{\kappa}$. Since $a \geq c(q^*)/\beta\kappa_\ell$ for all $a \in A^d(\beta\bar{\kappa})$ then $q_h = q_\ell = q^*$. If $A < c(q^*)/\beta\kappa_\ell$ then, from Lemma 13, (ω, a) is uniquely determined unless $(\phi - \beta\bar{\kappa})/\beta\kappa_\ell = i$ and $\pi_\ell \hat{\mathcal{S}}' [\kappa_\ell c(q^*)/\kappa_h] \geq i$ in which case $q_\ell = c^{-1}\left[\hat{\mathcal{S}}'^{-1}(i/\pi_\ell)\right]$ and $q_h = q^*$. (See proof of Lemma 13.) For given (ω, a) the problem (99)-(100) determines uniquely q_h and q_ℓ.

(iv) Suppose an equilibrium is nonmonetary. Then, $\omega(j) = 0$ and $a(j)$ is the unique solution to (105) for all $j \in \mathcal{B}$. Hence, $a(j) = A$. From (104) $\omega = 0$ requires

$$-i + \pi_h \hat{\mathcal{S}}'(\beta\kappa_h A) + \pi_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell A) \leq 0.$$

Define $i_0 = \pi_h \hat{\mathcal{S}}'(\beta\kappa_h A) + \pi_\ell \hat{\mathcal{S}}'(\beta\kappa_\ell A)$. By the contrapositive, if $i < i_0$ then the equilibrium is monetary. Provided that $\beta\kappa_\ell A < c(q^*)$, $\hat{\mathcal{S}}'(\beta\kappa_\ell A) > 0$ and $i_0 > 0$. Finally, if $\beta\kappa_\ell A \geq c(q^*)$ then $\phi = \beta\bar{\kappa}$ and $\omega = 0$. ■

If the economy-wide capital stock is large enough to allow agents to trade q^* in the PM for the lowest realization of κ then fiat money is not valued.[48] If the aggregate capital stock is too low relative to agents' liquidity needs (in terms of means of payment) then the price of capital increases above its fundamental value and fiat money can be valued provided that i is sufficiently low.

The expression for the price of capital in equilibrium is obtained from (102) by taking the first order condition for a, i.e.,

$$\phi = \beta\bar{\kappa} + \pi_h \beta\kappa_h \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \beta\kappa_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]. \tag{106}$$

[48] This result is in accordance with Lagos and Rocheteau (2006) who show that money is useful in the presence of capital in the Lagos-Wright environment if the first-best level of capital stock provides enough wealth for agents to trade in the PM, i.e., there is no shortage of capital to be used as means of payment.

It has two components. The first component is its fundamental value, $\beta\bar{\kappa}$. The second component is the liquidity value of capital in the PM, the last two terms on the right-hand side of (106). This liquidity value arises because capital can help relaxing buyers' budget constraint in a bilateral match.

To see how monetary policy can affect asset prices, take the first-order condition of (102) with respect to ω, and assume that the solution is interior (so that fiat money is valued),

$$i = \pi_h \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]. \tag{107}$$

As the cost of holding fiat money increases the marginal liquidity of wealth in the PM increases, which in turn raises ϕ.

The next proposition compares the (gross) rates of return of money and capital, $R_m = \gamma^{-1}$ and $R_a = \bar{\kappa}/\phi$, respectively. Let ρ denote the covariance between the return of capital, κ, and the marginal return of wealth in the PM, $[u'(q)/c'(q) - 1]$, i.e.,

$$\rho = \pi_h \left(\kappa_h - \bar{\kappa}\right) \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \left(\kappa_\ell - \bar{\kappa}\right) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]. \tag{108}$$

Proposition 12 *In any monetary equilibrium,*

$$R_a = \gamma^{-1} \left\{\frac{\rho}{\bar{\kappa}(1+i)} + 1\right\}^{-1} > R_m. \tag{109}$$

Proof. The expression for ϕ given by (106) can be rearranged as

$$\phi = \beta \left\{\bar{\kappa}(1+i) + \pi_h \left(\kappa_h - \bar{\kappa}\right) \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \left(\kappa_\ell - \bar{\kappa}\right) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]\right\}. \tag{110}$$

where I have used the fact that, from (104), $i = \left\{\pi_h \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] + \pi_\ell \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]\right\}$ in any monetary equilibrium. Substitute ρ by its expression given by (108) into (110) to get

$$\phi = \beta(1+i)\bar{\kappa} \left\{\frac{\rho}{\bar{\kappa}(1+i)} + 1\right\}.$$

Divide by $\bar{\kappa}$ and use the definition $\gamma = \beta(1+i)$ to get (109). In order to show that $R_a > R_m$ it is enough to establish that $\rho < 0$. Notice that $\pi_h \left(\kappa_h - \bar{\kappa}\right) + \pi_\ell \left(\kappa_\ell - \bar{\kappa}\right) = 0$. From (104), and since $\hat{\mathcal{S}}''(\beta\kappa a + \omega) < 0$ whenever $\hat{\mathcal{S}}'(\beta\kappa a + \omega) > 0$, in any monetary equilibrium $0 \leq \hat{\mathcal{S}}'(\beta\kappa_h a + \omega) < \hat{\mathcal{S}}'(\beta\kappa_\ell a + \omega)$. Since $\frac{u'(q_h)}{c'(q_h)} < \frac{u'(q_\ell)}{c'(q_\ell)}$ then

$$\pi_h \left(\kappa_h - \bar{\kappa}\right) \left[\frac{u'(q_h)}{c'(q_h)} - 1\right] < -\pi_\ell \left(\kappa_\ell - \bar{\kappa}\right) \left[\frac{u'(q_\ell)}{c'(q_\ell)} - 1\right]$$

and $\rho < 0$. ■

Capital has a higher rate of return than fiat money in any monetary equilibrium. This result holds even though agents are risk-neutral with respect to their AM consumption. This rate-of-return differential arises because capital is used as a means of payment in the PM where individuals are risk-averse. Capital yields a high dividend in matches when the marginal value of wealth in the PM is low, and a low dividend in matches where the marginal value of wealth is high.[49] In contrast, the value of money is constant and uncorrelated with the marginal utility of wealth in the PM. Finally, as $\kappa_h - \kappa_\ell \to 0$ then $\rho \to 0$ and $R_a = \gamma^{-1}$, i.e., money and capital have the same rate of return.

Incomplete information

I now describe succinctly the case where both buyers and sellers are uninformed about the future value of κ. Buyers choose their portfolios in order to maximize $-i\omega - (\phi - \beta\bar{\kappa})\, a + \hat{\mathcal{S}}(\omega + \beta\bar{\kappa}a)$. If $A < c(q^*)/\beta\bar{\kappa}$ then $\phi > \beta\bar{\kappa}$ and there is $i_0 > 0$ such that for all $i < i_0$ an equilibrium is monetary. Moreover, if a monetary equilibrium exists then $\phi = \beta\bar{\kappa}(1+i)$ and $1+i = u'(q)/c'(q)$ where q is the quantity produced and consumed in bilateral matches in the PM. In this case, $R_a = R_m$, i.e., fiat money and capital have the same rate of return.

[49] This result is analogous to the one in Lagos (2006) who finds that even in the absence of legal restrictions on the use of assets as means of payment his model can be consistent with an equity-premium puzzle, i.e., a too large return differential between bonds and equity.

F. Endogenizing sellers' portfolio choices

This appendix considers the model in Section 6 where both the real asset and fiat money are traded in the AM, and it relaxes the assumption that sellers cannot produce in the AM (and hence cannot accumulate assets.) The utility function of a seller becomes

$$U_t^s = -\ell_t - c(q_t) + \beta x_{t+1}$$

where $\ell_t \in \mathbb{R}_+$ is the disutility of effort of the seller. The production technology in the AM is linear ($y_t = \ell_t$). I assume that both the buyer's and the seller's portfolios are common knowledge in a match in the PM, and I maintain the assumption that buyers have some private information about the future value of the dividend.

Consider first the bargaining problem in the PM. The buyer's portfolio is denoted by $\left(\omega^b, a^b\right)$ while the seller's portfolio is (ω^s, a^s). The offer made by a buyer with private information $\kappa_j \in \{\kappa_\ell, \kappa_h\}$ solves:

$$\max_{q,d,\tau} \left[u(q) - \beta\kappa_j d - \frac{\beta}{\gamma}\tau\right] \tag{111}$$

$$\text{s.t.} \quad -c(q) + \lambda(q,d,\tau)\beta\kappa_h d + [1-\lambda(q,d,\tau)]\,\beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0 \tag{112}$$

$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b. \tag{113}$$

The novelty with respect to the model in Section 6 is the constraint (113) according to which the seller holds some assets, which he can transfer to the buyer.

The following Lemma rules out pooling offers.

Lemma 14 *In equilibrium, there is no pooling offer such that $d \neq 0$.*

Proof. Suppose first that there is a pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ such that $\bar{d} > 0$. One can follow the proof of Lemma 5 to show that such an equilibrium would violate the Intuitive Criterion. Suppose next that the pooling offer is such that $\bar{d} < 0$, i.e., the seller transfers some of his real asset to the buyer. The participation constraint of the seller implies $\bar{\tau} > 0$. Since $\bar{d} < 0$ and $\lambda(\bar{q}, \bar{d}, \bar{\tau}) \in (0,1)$ then

$$-c(\bar{q}) + \lambda(\bar{q}, \bar{d}, \bar{\tau})\beta\kappa_h\bar{d} + \left[1 - \lambda(\bar{q}, \bar{d}, \bar{\tau})\right]\beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau} < -c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}.$$

Hence, a necessary condition for the pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ to be acceptable is

$$-c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau} > 0.$$

I will establish that there is an unsent offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ such that

$$u(\tilde{q}) - \beta\kappa_h\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad < \quad \bar{S}^h \tag{114}$$

$$u(\tilde{q}) - \beta\kappa_\ell\tilde{d} - \frac{\beta}{\gamma}\tilde{\tau} \quad > \quad \bar{S}^\ell \tag{115}$$

$$-c(\tilde{q}) + \beta\kappa_\ell\tilde{d} + \frac{\beta}{\gamma}\tilde{\tau} \quad \geq \quad 0, \tag{116}$$

where $\bar{S}^h \equiv u(\bar{q}) - \beta\kappa_h\bar{d} - \frac{\beta}{\gamma}\bar{\tau}$ and $\bar{S}^\ell \equiv u(\bar{q}) - \beta\kappa_\ell\bar{d} - \frac{\beta}{\gamma}\bar{\tau}$. According to (114), the offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ would make a buyer in the high-dividend state worse-off relative to his equilibrium payoff; from (115), it would make a buyer in the low-dividend state better off; from (116), it would be accepted by sellers provided that they believe that $\kappa = \kappa_\ell$. If one can find such an offer then the proposed equilibrium fails the Intuitive Criterion.

Consider an out-of-equilibrium offer such that $\tilde{q} = \bar{q}$ and $\tilde{d} = \bar{d} + \varepsilon$ where $\varepsilon > 0$. The conditions (114) and (115) imply

$$\kappa_\ell < \frac{\bar{\tau} - \tilde{\tau}}{\varepsilon\gamma} < \kappa_h. \tag{117}$$

The condition (116) requires then

$$\beta\varepsilon\left[\left(\frac{\bar{\tau} - \tilde{\tau}}{\gamma\varepsilon}\right) - \kappa_\ell\right] \leq -c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}. \tag{118}$$

From (117),

$$\beta\varepsilon\left[\left(\frac{\bar{\tau} - \tilde{\tau}}{\gamma\varepsilon}\right) - \kappa_\ell\right] < \beta\varepsilon(\kappa_h - \kappa_\ell).$$

Hence, for any $\varepsilon > 0$ such that $\varepsilon \leq \left[-c(\bar{q}) + \beta\kappa_\ell\bar{d} + \frac{\beta}{\gamma}\bar{\tau}\right]/\beta(\kappa_h - \kappa_\ell)$ the inequality (118) is satisfied. For any ε, one can find $\tilde{\tau}$ such that (117) holds. Provided that ε is sufficiently small, the offer $(\tilde{q}, \tilde{d}, \tilde{\tau})$ is feasible. Consequently, the proposed equilibrium with the pooling offer $(\bar{q}, \bar{d}, \bar{\tau})$ violates the Intuitive Criterion. ■

From Lemma 14, the offer made by a buyer in the low-dividend state must satisfy $-c(q) + \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0$. Hence, the buyer's payoff in the low-dividend state is bounded above by his complete information payoff. The complete information offer solves

$$(q_\ell, d_\ell, \tau_\ell) = \arg\max_{q,d,\tau}\left[u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau\right]$$
$$\text{s.t.} \quad -c(q) + \beta\kappa_\ell d + \frac{\beta}{\gamma}\tau \geq 0$$
$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b.$$

The solution is $q_\ell = q^*$ and $\beta\kappa_\ell d_\ell + \frac{\beta}{\gamma}\tau_\ell = c(q^*)$ if $\beta\kappa_\ell a^b + \omega^b \geq c(q^*)$. Otherwise, $\frac{\beta}{\gamma}\tau_\ell = \omega^b$, $d_\ell = a^b$ and $c(q_\ell) = \beta\kappa_\ell a^b + \omega^b$. Hence, there is always a complete-information offer such that $d_\ell \geq 0$. Since $\lambda(q,d,\tau)\beta\kappa_h d + [1-\lambda(q,d,\tau)]\,\beta\kappa_\ell d \geq \beta\kappa_\ell d$ for all $d \geq 0$, the complete information offer is acceptable for any belief system.

Consider next the offer made by a buyer in the high-dividend state. From Lemma 14, it satisfies $-c(q) + \beta\kappa_h d + \frac{\beta}{\gamma}\tau \geq 0$. If $\omega^b \geq c(q^*)$ then the buyer can achieve his complete-information payoff by offering $q_h = q^*$, $d_h = 0$ and $\frac{\beta}{\gamma}\tau_h = c(q^*)$. Provided that $\omega^b < c(q^*)$, the buyer cannot make the complete information offer since otherwise it would be imitated by a buyer in the low-dividend state. In this case, the buyer makes an offer that maximizes his payoff but that does not provide a buyer in the low-dividend state with strict incentives to imitate it. So, the offer solves

$$(q_h, d_h, \tau_h) = \arg\max_{q,d,\tau}\left[u(q) - \beta\kappa_h d - \frac{\beta}{\gamma}\tau\right]$$
$$\text{s.t.} \quad -c(q) + \beta\kappa_h d + \frac{\beta}{\gamma}\tau \geq 0$$
$$u(q) - \beta\kappa_\ell d - \frac{\beta}{\gamma}\tau \leq \bar{S}^\ell$$
$$-\omega^s \leq \frac{\beta}{\gamma}\tau \leq \omega^b, \quad -a^s \leq d \leq a^b.$$

The solution to this problem is similar to the one described in Lemma 6. The incentive-compatibility constraint cannot be slack since otherwise (q_h, d_h, τ_h) would coincide with the complete information offer. Suppose that the seller's participation constraint is slack. Then, the incentive-compatibility constraint is binding which implies

$$u(q_h) - \beta\kappa_h d_h - \frac{\beta}{\gamma}\tau_h = \bar{S}^\ell - (\kappa_h - \kappa_\ell)\,\beta d_h,$$

and hence

$$\bar{S}^h = \beta \max_{-a^s \leq d \leq a^b} (\kappa_\ell - \kappa_h)\, d + \bar{S}^\ell = \beta\,(\kappa_h - \kappa_\ell)\, a^s + \bar{S}^\ell.$$

The buyer asks for the whole stock of real assets held by the seller. Using the fact that $d_h = -a_s$, the highest achievable payoff for the buyer is

$$\bar{S}^h = \max_{q,d,\tau}\,[u(q) - z]$$
$$\text{s.t.} \quad -c(q) + z \geq 0$$
$$-\omega^s - \beta\kappa_h a^s \leq z \leq \omega^b - \beta\kappa_h a^s.$$

Provided that $\omega^b < c(q^*)$ and $a^b > 0$, it can be checked that $\bar{S}^\ell > \bar{S}^h$ which is a contradiction. So, the seller's participation constraint is binding.

Since sellers get no surplus in the PM trades, their problem in the AM is

$$\max_{\omega,a} \left\{-i\omega - (\phi - \beta\bar{\kappa})\, a\right\}.$$

They accumulate no real balances ($\omega^s = 0$), and they hold the real asset ($a^s > 0$) only if $\phi = \beta\bar{\kappa}$. Sellers hold an asset only if it is priced at its fundamental value.

G. Endogenous capital stock

The model of Section 6 can be readily extended to endogenize the quantity of real assets in the economy. I lay down succinctly such an extension. The approach is similar to the one in Lagos and Rocheteau (2006).

Suppose that buyers can produce capital goods in the first period of their lives. The disutility cost to produce a units of capital is $\psi(a)$ with $\psi(0) = 0$, $\psi' > 0$ and $\psi'' > 0$. Capital goods are one-period lived, and they generate $\kappa \in \{\kappa_\ell, \kappa_h\}$ units of AM-goods in the subsequent period.

The problem of a young buyer in the AM is

$$\max_{\omega,a} \left\{-i\omega - \psi(a) + \pi_h S^h(\omega, a) + \pi_\ell S^\ell(\omega, a) + \beta\bar{\kappa}a\right\}.$$

The determination of the terms of trade in the PM is characterized by Lemmas 5 and 6. Following the proof of Lemma 7, it can be shown that there is a unique solution (ω, a) to the buyer's problem, and hence equilibrium is unique.

Let a^* denote the socially efficient level of capital, i.e., the solution to $\psi'(a) = \beta\bar{\kappa}$. If $a^* \geq c(q^*)/\beta\kappa_\ell$ then $a = a^*$ and $q_\ell = q^*$. Buyers have enough wealth to buy the first best quantity of output in the low-dividend state. A monetary equilibrium exists provided that $i < i_0(a^*)$ where i_0 is defined by

$$i_0 = \pi_h S^h_\omega(0, a^*) = \pi_h \Delta(q_h) \left(1 - \frac{\kappa_\ell}{\kappa_h}\right),$$

where q_h is solution to (48)-(49) with $\omega = 0$. So, in contrast to Lagos and Rocheteau (2006), there exists a monetary equilibrium for any level of a^*.

If $a^* < c(q^*)/\beta\kappa_\ell$ then there exists a threshold $\tilde{\imath}$ defined as

$$\tilde{\imath} = \pi_h S^h_\omega \left(c(q^*) - \beta\kappa_\ell a^*, a^*\right)$$

such that if $i > \tilde{\imath}$ then $a > a^*$. Buyers overaccumulate capital because of the liquidity services it provides in the PM. Denote $\hat{a}$ the solution to $\psi'(a) = \pi_h S^h_a(0, a) + \pi_\ell S^\ell_a(0, a) + \beta\bar{\kappa}$. A monetary equilibrium exists provided that $i < i_0(\hat{a})$ where

$$i_0(\hat{a}) = \pi_h S^h_\omega(0, \hat{a}) + \pi_\ell S^\ell_\omega(0, \hat{a})$$

Following a similar argument as in Proposition 6, it can be shown that if $a^* < c(q^*)/\beta\kappa_\ell$ and $i \in (\tilde{\imath}, i_0(\hat{a}))$ then $da/di > 0$. An increase in inflation induces buyers to accumulate more capital.

Finally, the rate of return of capital is $R_a = \bar{\kappa}/\psi'(a)$. Following the proof of Proposition 7, it can be shown that in any monetary equilibrium $R_a > R_m$, capital dominates fiat money in its rate of return. In contrast, in Lagos and Rocheteau (2006), fiat money and capital have the same rate of return in any monetary equilibrium. Moreover, if $i < \tilde{\imath}$ then $\psi'(a) = \beta\bar{\kappa}$ so that $R_a = \beta^{-1}$, the rate of return of capital is equal to the gross discount rate.